Classification of PDEs, Method of Characteristics, Traffic Flow Problem

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1 Background

1.1 Linear PDEs

Classification of linear PDE of 2nd order in two independent variables

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + F u + G = 0, \]

where \( A, B, ..., G \) are constant coefficients.

Three categories of PDEs can be distinguished:

\[ B^2 - 4AC < 0 \quad \text{elliptic PDE} \]

\[ B^2 = 4AC \quad \text{parabolic PDE} \]

\[ B^2 - 4AC > 0 \quad \text{hyperbolic PDE} \]

Classification depends only on the highest-order derivatives in each independent variables.

If \( A, B, ..., G \) are functions of \( x, y, u, u_x \) or \( u_y \) the classification still can be used with local interpretation.
Well-posed mathematical problem

• the solution exist
• the solution is unique
• the solution depends continuously on the auxiliary (IC/BC) data

Well-posed computational problem

• the computational solution exist
• the computational solution is unique
• the computational solution depends continuously on the approximate auxiliary data

BC & IC

Notation: If computational domain $R$, boundary $\partial R$, normal to boundary $\mathbf{n}$, tangential to boundary $\mathbf{s}$.

• Dirichlet condition, e.g. $u = f$ on $R$

• Neumann (derivative) condition, e.g. $rac{\partial u}{\partial n} = f$ or $rac{\partial u}{\partial s} = g$ on $\partial R$

• mixed or Robin condition, e.g. $rac{\partial u}{\partial n} + ku = f$, $k > 0$ on $\partial R$
1.2 Classification

Classification by characteristics

- For a single 1st-order hPDE w. two independent variables,

\[ A \frac{\partial u}{\partial t} + B \frac{\partial u}{\partial x} = C \]  

(2)

a single real characteristics exists through ∀ point, and the characteristic direction is defined

\[ \frac{dx}{dt} = \frac{A}{B} \]  

(3)

Along the characteristics directions Eq. (2) reduces to

\[ \frac{du}{dt} = \frac{C}{A} \quad \& \quad \frac{du}{dx} = \frac{C}{B} \]  

(4)

Eq. (4) can be integrated as ODE along the grid defined by Eq. (3) provided the initial data are given on a non-characteristics line.
Concept of characteristic directions for a 2nd-order PDE w. two independent variables can be established. Since only highest derivatives determine the category of PDE, Eq. (1) can be written as

\[ A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + H = 0, \]  

where \( H \) contains all other terms. It is possible to obtain \( \forall \in \mathbb{R} \) two directions along which the integration of Eq. (5) involves only two total differentials. The existence of these (characteristics) directions relates directly to the category of PDE.

Introduce

\[ P = \frac{\partial u}{\partial x}, Q = \frac{\partial u}{\partial y}, R = \frac{\partial^2 u}{\partial x^2}, S = \frac{\partial^2 u}{\partial x \partial y}, T = \frac{\partial^2 u}{\partial y^2}. \]

Further, a curve \( K \) is introduced in \( \mathbb{R} \) on which Eq. (5) is satisfied. Along a tangent to \( K \)

\[ dP = Rdx + Sdy \quad \& \quad dQ = Sdx + Tdy, \]

where \( dy/dx \) defines the slope of tangent to \( K \), and, Eq (5) can be written as

\[ AR + BS + CT + H = 0. \]  

(6)
I Classification of PDEs

Eq. (6) can be written as

\[
S \left[ A \left( \frac{dy}{dx} \right)^2 - B \left( \frac{dy}{dx} \right) + C \right] - \left\{ A \left( \frac{dP}{dx} \right) + H \right\} \frac{dy}{dx} + C \frac{dQ}{dx} = 0 \quad (7)
\]

If \( dy/dx \) is chosen such that

\[
A \left( \frac{dy}{dx} \right)^2 - B \left( \frac{dy}{dx} \right) + C = 0 \quad (8)
\]

the solutions to Eq. (8) define the characteristic directions.

**Conclusion:**

- for a hPDE two real characteristics exist
- for a pPDE one real characteristic exists
- for an ePDE the characteristics are complex

The discriminant \( B^2 - 4AC \) determines both type of PDE and nature of characteristics.
I Classification of PDEs

- System of equations

Two-component system of 1\textsuperscript{st} order PDE

\[
\begin{align*}
A_{11} \frac{\partial u}{\partial x} &+ B_{11} \frac{\partial u}{\partial y} + A_{12} \frac{\partial v}{\partial x} + B_{12} \frac{\partial v}{\partial y} = E_1 \quad (9) \\
A_{21} \frac{\partial u}{\partial x} &+ B_{21} \frac{\partial u}{\partial y} + A_{22} \frac{\partial v}{\partial x} + B_{22} \frac{\partial v}{\partial y} = E_2 \quad (10)
\end{align*}
\]

After re-arranging Eqs (9)-(10)

\[
\begin{bmatrix}
(A_{11} dy - B_{11} dx) & (A_{21} dy - B_{21} dx) \\
(A_{12} dy - B_{12} dx) & (A_{22} dy - B_{22} dx)
\end{bmatrix}
\begin{bmatrix}
L_1 \\
L_2
\end{bmatrix} = 0
\]

(11)

where \(L_1, L_2\) are suitable multipliers. Since the system is homogeneous in \(L_i\), it is necessary that

\[
\text{det}[A dy - B dx] = 0.
\]

(12)
I Classification of PDEs

For non-trivial solution, i.e. Eq. (12) gives

\[
\text{DIS} = (A_{11}B_{22} - A_{21}B_{12} + A_{22}B_{11} - A_{12}B_{21})^2 \\
-4(A_{11}A_{22} - A_{21}A_{12})(B_{11}B_{22} - B_{21}B_{12})
\]  

(13)

Classification:

<table>
<thead>
<tr>
<th>DIS</th>
<th>Roots</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive</td>
<td>2 real</td>
<td>hyperbolic</td>
</tr>
<tr>
<td>zero</td>
<td>1 real</td>
<td>parabolic</td>
</tr>
<tr>
<td>negative</td>
<td>2 complex</td>
<td>elliptic</td>
</tr>
</tbody>
</table>
I Classification of PDEs

- System of $n$ 1st order PDEs

\[
\begin{bmatrix}
A \left( \frac{\partial y}{\partial x} \right)^{(k)} - B \\
\end{bmatrix} L^{(k)} = 0, \quad k = 1, \ldots, n \quad (14)
\]

where $L_i$ are suitable multipliers.

**Classification:**

The character of the system depends on the solution to Eq. (12):

- If $n$ real roots $\rightarrow$ system is hyperbolic
- If $\nu$ real roots, $1 \leq \nu \leq n - 1$, and $\nexists$ complex roots $\rightarrow$ system is parabolic
- If $\nexists$ real roots $\rightarrow$ system is elliptic.

**Note:** Most important whether ePDE or non-ePDE, since latter preclude time-like behavior.
Classification of PDEs

Classification by Fourier Analysis

Useful

- if higher-order derivatives appear
- since can indicate the expected behavior of solution (oscillatory, exp. growth, etc.)

Example

\[
A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0,
\]  

(15)

Introduce

\[
\hat{u} = \mathcal{F} u
\]

where \( \mathcal{F} \) is the Fourier transform defined by

\[
\hat{u} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, y) \exp(-i\sigma_x x) \exp(-i\sigma_y y) dx dy
\]

(16)

Eq. (15) transforms into

\[
[A(i\sigma_x)^2 + B(i\sigma_x i\sigma_y) + C(i\sigma_y)^2] \hat{u}
\]

(17)

often called the characteristic polynomial or the symbol of the PDE.
I Classification of PDEs

Wave equation

Simplest hPDE is wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{18}
\]

For

IC: \( u(x, 0) = \sin \pi x, \frac{\partial u}{\partial t}(x, 0) = 0 \)

and

BC: \( u(0, t) = u(1, t) = 0 \Rightarrow u(x, t) = \sin \pi x \cos \pi t \)

Note: lack of attenuation is feature of linear hPDE. hPDEs produce real characteristics. E.g. for wave equation Eq. (18) characteristics directions are given by

\( \frac{dx}{dy} = \pm 1. \)

see Fig. 1
Figure 1: Characteristics or the wave equation
Wave representation

Q: Whether the discretisation process represents waves of short or long wavelength with the same accuracy?

- Significance of grid coarseness:

  - FDM replaces a continuous func \( g(x) \rightarrow \) with vector of nodal values \((g_j)\).
  - Choice of grid spacing \( \Delta x \) depends on smoothness of \( g(x) \) (Fig. 2a) poor choice, Fig. 2b reasonable choice)
A $\mathcal{F}$-rep of $g(x)$

$$g(x) = \sum_{m=-\infty}^{\infty} g_m e^{imx} \quad (19)$$

$g_m$, amplitude of Fourier mode of wavelength $\lambda = 2\pi/m$,

$$g_m = \frac{1}{2\pi} \int_{0}^{2\pi} g(x)e^{-imx} \, dx. \quad (20)$$

A $\mathcal{F}$-rep of vector of nodal ($g_j$)

$$g_j = \sum_{m=1}^{J} g_m e^{imj\Delta x} \quad (21)$$

where the modal amplitude $g_j$ is given

$$g_m = \Delta x \sum_{j=1}^{J} g_j e^{-imj\Delta x}. \quad (22)$$

Wavelength shorter than cut-off wavelength $\lambda = 2\Delta x$ cannot be represented. ⇒ ($g_j$) should be interpreted as a long-wave approx of $g(x)$. 
Accuracy of representing waves

Accuracy may be assessed by comparing FD approximate solutions to progressive waves

\[ \bar{T}(x, t) = \Re \{ e^{im(x-qt)} \} = \cos[m(x - qt)]. \quad (23) \]

At \((j, n)\)-th node the exact values of 1st/2nd derivatives of \( \bar{T} \):

\[ \frac{\partial \bar{T}}{\partial x} = -m \sin[m(x_j - rt_j)] \quad (24) \]

\[ \frac{\partial^2 \bar{T}}{\partial x^2} = -m^2 \cos[m(x_j - rt_j)] \quad (25) \]

- Substitution into three-point formula (3PT)

\[ \left[ \frac{\partial \bar{T}}{\partial x} \right]_j^n \approx \frac{(T_{j+1}^n - T_{j-1}^n)}{2\Delta x} \]

and calculating the amp ratio of 1st derivatives gives

\[ AR(1)_{3PT} = \frac{[\partial \bar{T} \partial x]_j^n}{[\partial T \partial x]} = \frac{\sin(m\Delta x)}{m\Delta x}. \quad (26) \]
**I Classification of PDEs**

- Using CD approx

\[
\left( \frac{T_{j-1}^n - 2T_j^n + T_{j+1}^n}{\Delta x^2} \right)
\]
gives the amp ratio for 2nd derivatives

\[
AR(2)_{3PT} = \left( \frac{\sin(m\Delta x/2)}{m\Delta x/2} \right)^2.
\] (27)

**Summary table:**

<table>
<thead>
<tr>
<th>Derivative</th>
<th>Amplitude ratio LW ($\lambda = 20\Delta x$)</th>
<th>Amplitude ratio SW ($\lambda = 4\Delta x$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{dT}{dx}$</td>
<td>0.64</td>
<td>0.984</td>
</tr>
<tr>
<td>$\frac{d^2T}{dx^2}$</td>
<td>0.992</td>
<td>0.405</td>
</tr>
</tbody>
</table>
2 Quasi-linear first-order PDEs

We consider only two independent variables \((x, y)\) and an unknown \(z = z(x, y)\) satisfying a first order PDE. This PDE is quasi-linear if it is linear in its highest order terms, i.e.

\[
\begin{align*}
  z_x &= \frac{\partial z}{\partial x} \quad \text{and} \quad z_y = \frac{\partial z}{\partial y}.
\end{align*}
\]

Thus

\[
\begin{align*}
  zz_x + z_y &= 0 \quad \text{is quasi-linear (and non-linear)} \\
  (z_x)^2 + z_y &= 0 \quad \text{is not quasi-linear}.
\end{align*}
\]

The most general first-order quasi-linear PDE is:

\[
P z_x + Q z_y = R \quad (28)
\]

where

\[
P = P(x, y, z), \quad Q = Q(x, y, z), \quad R = R(x, y, z) \quad (29)
\]

are given continuous functions.
Method of Characteristics

Consider the family of curves in the \((x, y)\) plane satisfying

\[
\frac{dy}{dx} = \frac{Q}{P} \quad \text{or} \quad \frac{dx}{dy} = \frac{P}{Q} \quad \text{or} \quad \frac{dx}{P} = \frac{dy}{Q}.
\]

Suppose \(z\) is known at a point \(A(x, y)\). There is one curve \(\Gamma_A\) of this family through \(A\), and along \(\Gamma_A\)

\[
dz = z_x dx + z_y dy = \left(z_x + \frac{Q}{P} z_y\right) dx = \frac{R}{P} dx \quad (30)
\]

using Eq. (28). Hence \(\frac{dz}{dx} = \frac{R}{P}\) along \(\Gamma_A\), and so:

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \quad (31)
\]

Eqs. (31) are known as the associated equations for Eq. (28), and are equivalent to Eq. (28). For let each term in Eq. (31) be \(ds\), so \(dx = Pds\), \(dy = Qds\), \(dz = Rds\). Substitute in Eq. (30) to get

\[
Rds = Pz_x ds + Qz_y ds \Rightarrow Pz_x + Qz_y = R,
\]

i.e. Eq. (28).
• Suppose $z$ is given along a curve $C$ in the $(x, y)$ plane. Through each point $A$ on $C$, we can continue the solution along $\Gamma_A$ in both directions provided $\Gamma_A$ is not parallel to $C$, i.e. provided that, on $C$, $\frac{dy}{dx}$ is nowhere equal to $\frac{Q}{P}$. The curves $\Gamma_A$ are known as the characteristics. Provided the characteristics do not intersect, we obtain a region bounded by $\Gamma_+$ and $\Gamma_-$ within which $z$ is known. If $\frac{Q}{P}$ is independent of $z$ the characteristics are independent of the boundary conditions. In particular $\frac{Q}{P}$ is independent of $z$ for a linear PDE. If $P$ and $Q$ are constants, then the characteristics are parallel straight lines.
Example 1

Solve $z_x - z_y = 1$ with $z = x^2$ on $y = 0$.

Solution

The associated equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{1} \Rightarrow \frac{dy}{dx} = -1, \quad \frac{dz}{dx} = 1.$$ 

Thus the characteristics are $x + y = \alpha$ and on the characteristics $z - x = \beta$.

Figure 4: Characteristics of Example 1
The curve $C$ is $y = 0$ and each point on $C$ is intercepted by exactly one characteristic. We can proceed in two ways.

(A) When $y = 0$, $x + y = \alpha \Rightarrow x = \alpha$ and $z = \alpha^2$.

Hence from $z = x + \beta \Rightarrow \beta = \alpha^2 - \alpha$.

Thus $z = x + (\alpha^2 - \alpha)$ on $x + y = \alpha$.

Eliminate $\alpha$ to get $z = x + ((x + y)^2 - (x + y)) \Rightarrow$

$$z = (x + y)^2 - y.$$ 

(B) Since $z - x$ is constant when $x + y$ is constant $\Rightarrow$

$z - x = f(x + y)$ for some function $f$. But $z = x^2$ when $y = 0 \Rightarrow x^2 - x = f(x)$.

Thus $z = x + ((x + y)^2 - (x + y)) \Rightarrow$

$$z = (x + y)^2 - y.$$
Example 2

Solve $yz_x + xz_y = z$ with $z = x^3$ on $y = 0$ and $z = y^3$ on $x = 0$.

Solution

The associated equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z} \Rightarrow$$

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow x^2 - y^2 = \alpha$$

are characteristics, where $\alpha =$ const. Then

$$\frac{dz}{dx} = \frac{z}{\sqrt{(x^2 - \alpha)}} \Rightarrow \frac{dz}{z} = \frac{dx}{\sqrt{(x^2 - \alpha)}}$$

$$\Rightarrow$$

$$\ln z = \ln \left(x + \sqrt{(x^2 - \alpha)}\right) + \beta'$$

$$z = \beta \left(x + \sqrt{(x^2 - \alpha)}\right)$$

on a characteristic, where $\beta = e^{\beta'} =$ const.

$$\alpha = x^2 - y^2 \Rightarrow z = \beta \left(x + \sqrt{(x^2 - x^2 + y^2)}\right)$$

$$= \beta(x + y) \text{ or } \beta(x - y).$$
Case 1: $z = \beta(x + y)$

In this case we find, that as in Ex 1 (B) above,

$$\frac{z}{x + y} \text{ is constant when } x^2 - y^2 \text{ is constant.}$$

The GS is therefore $z = (x + y)f(x^2 - y^2)$, and it remains to determine $f$.

We are given $z$ on both axes. At the common point $O$, $z = 0$ from both prescriptions. The characteristics through $O$ are $x = \pm y$ ($\alpha = 0$) and on these $z = 0$. The result is obviously symmetric about both axes.
II Method of Characteristics

Suppose $\alpha > 0$.

Consider $A(\alpha_1^{\frac{1}{2}}, 0)$ at which

$$z = x^3 = \alpha_1^{\frac{3}{2}}.$$  

So from the GS $\Rightarrow$

$$\alpha_1^{\frac{3}{2}} = \alpha_1^{\frac{1}{2}} f(\alpha_1) \Rightarrow f(\alpha_1) = \alpha_1 \Rightarrow z = (x + y)(x^2 - y^2)$$  

for $x^2 > y^2$.

Suppose $\alpha < 0$.

Consider $B(0, (-\alpha_2)^{\frac{1}{2}})$ at which

$$z = y^3 = (-\alpha_2)^{\frac{3}{2}}.$$  

So from the GS $\Rightarrow$

$$(-\alpha_2)^{\frac{3}{2}} = (-\alpha_2)^{\frac{1}{2}} f(\alpha_2) \Rightarrow f(\alpha_2) = -\alpha_2 \Rightarrow z = (x + y)(y^2 - x^2)$$  

for $x^2 < y^2$.

In summary

$$z = \begin{cases}  
(x + y)(x^2 - y^2) & \text{for } x^2 > y^2 \\
0 & \text{for } x^2 = y^2 \\
(x + y)(y^2 - x^2) & \text{for } x^2 < y^2 
\end{cases}$$  \hspace{1cm} (32)
**Method of Characteristics**

Case 2: $z = \beta(x - y)$

In this case it can be similarly deduced that the GS of the PDE is

$$z = (x - y)g(x^2 - y^2).$$

However, this GS gives

$$z_x = g(x^2 - y^2) + 2x(x - y)g'(x^2 - y^2)$$

$$z_y = -g(x^2 - y^2) - 2y(x - y)g'(x^2 - y^2)$$

Therefore

$$yz_x + xz_y = -(x - y)g(x^2 - y^2) = -z$$

which is not our original PDE, therefore we dismiss this second case, $z = \beta(x - y)$, as spurious solution.
We begin by considering Eq. (32). It is clear that $z$ is everywhere continuous, and that $z_x, z_y$ are everywhere continuous except possibly on the lines

$$x = \pm y \quad (x^2 - y^2) = 0.$$ 

From Eq. (32) we find

$$x^2 > y^2 : \quad z_x = (x^2 - y^2) + 2x(x + y) = (x + y)(3x - y)$$
$$z_y = (x^2 - y^2) - 2y(x + y) = (x + y)(x - 3y)$$

$$x^2 < y^2 : \quad z_x = (y^2 - x^2) - 2x(x + y) = (x + y)(y - 3x)$$
$$z_y = (y^2 - x^2) + 2y(x + y) = (x + y)(3y - x).$$

Thus as $x \to -y$ from either side, $z_x \to 0$ and $z_y \to 0$. Hence $z_x$ and $z_y$ are continuous on $x + y = 0$.

However, as $x \to y$, $z_x$ jumps from $+4x^2 \ (x > y)$ to $-4x^2 \ (x < y)$, and $z_y$ jumps from $-4x^2 \ (x > y)$ to $+4x^2 \ (x < y)$. Thus $z_x$ and $z_y$ are discontinuous across the characteristic $x = y$. 
2.1 Some properties of characteristics

We investigate the possibility of discontinuities in $z_x$ and $z_y$ for Eq.(28), but we shall suppose $z$ is everywhere continuous. (The standard terminology is that we are looking for weak discontinuities whereas discontinuities in $z$ itself are strong discontinuities).

Suppose $C$ is approached from + and −. Then

\[
\begin{align*}
\delta z^+ &= \frac{\partial z^+}{\partial x} \delta x + \frac{\partial z^+}{\partial y} \delta y \\
\delta z^- &= \frac{\partial z^-}{\partial x} \delta x + \frac{\partial z^-}{\partial y} \delta y
\end{align*}
\]

where $(\delta x, \delta y)$ is along $C$. Subtract.
II Method of Characteristics

• Weak discontinuities

Because $z$ is continuous, $\delta z^+ = \delta z^-$. Hence

$$\delta x \left[ \frac{\partial z}{\partial x} \right]^+ + \delta y \left[ \frac{\partial z}{\partial y} \right]^+ = 0. \quad (33)$$

where the square brackets denote the jump in the expression across $C$.

Since Eq. (28) is satisfied on both sides and since, by hypothesis, $P$, $Q$, $R$ are continuous

$$P \left[ \frac{\partial z}{\partial x} \right]^+ + Q \left[ \frac{\partial z}{\partial y} \right]^+ = 0. \quad (34)$$

The necessary condition for

$$\left[ \frac{\partial z}{\partial x} \right]^- \neq 0 \quad \text{and} \quad \left[ \frac{\partial z}{\partial y} \right]^- \neq 0 \quad \text{is} \quad \frac{\delta x}{P} = \frac{\delta y}{Q},$$

i.e.

$$\frac{dx}{P} = \frac{dy}{Q}. \quad (35)$$

Thus $C$ must be a characteristic $\Gamma_A$. 
II Method of Characteristics

- When $Q/P$ is independent of $z$, the characteristics are independent of the boundary conditions.

When $Q/P$ depends on $z$, different boundary conditions produce different sets of characteristics.

![Figure 7: (i) $z$ determined throughout rectangle; (ii) $z$ not determined in $BCD$](image)

- **Strong discontinuities**

We can also consider situations in which $z$ itself is discontinuous at a point on the boundary. Then the shape of the characteristics (in the case when $Q/P$ depends on $z$) will change discontinuously at that point. Qualitatively, there are two possibilities:
II Method of Characteristics

- In (i) there appear to be **two** characteristics through a point, whereas
- in (ii) there is a region containing **no** characteristics.

![Characteristics](image)

Figure 8: Characteristics leading to (i) shocks; or to (ii) centred fans (also called rarefaction shocks)

Again, qualitatively these two situations will be relevant to our models of traffic flow [(i) leads to shocks, (ii) leads to centred fans - see later].

- We can obtain similar situations to (i) and (ii), but even **without** initial discontinuities in the slopes of the characteristics. The following example will connect well with our **models of traffic flow**.
II Method of Characteristics

Example

Solve

$$\rho_t + \rho \rho_x = 0$$

with $\rho = f(x)$ on $t = 0$. Consider two special cases:

**Case 1:** $f(x) = 0$ ($x < 0$), $f(x) = x$ ($0 \leq x < 1$), $f(x) = 1$ ($x \geq 1$);

**Case 2:** $f(x) = 0$ ($x < 0$), $f(x) = -x$ ($0 \leq x < 1$), $f(x) = -1$ ($x \geq 1$).

The associated equations Eq. (31) are

$$\frac{dt}{1} = \frac{dx}{\rho} = \frac{d\rho}{0}$$

where the last is to be interpreted as $d\rho = 0$.

Thus

$$\rho = \alpha, \quad \frac{dx}{dt} = \alpha \Rightarrow x - \alpha t = \beta \quad \text{are characteristics.}$$
Now consider the characteristic through \( x = \xi \) on \( t = 0 \). On this characteristic \( \rho = \alpha = f(\xi) \). Thus

\[
x = f(\xi)t + \xi, \quad \rho = f(\xi)
\]  

(36)

![Figure 9: Characteristics for Example](image)

Alternatively, from Eq. (36)

\[
x = \rho t + \xi
\]

so that we can write Eq. (36) implicitly as

\[
\rho = f(x - \rho t)
\]  

(37)
**II Method of Characteristics**

*Case 1*: From Eq. (37)

\[
\rho = 0 \quad (x < 0), \quad \rho = x - \rho t \quad (0 \leq x - \rho t < 1)
\]

\[\Rightarrow\]

\[
\rho = \frac{x}{1 + t} \quad (0 \leq x < 1 + t), \quad \rho = 1 \quad (x \geq 1 + t),
\]

i.e.

\[
\rho = \begin{cases} 
0 & (x < 0) \\
\frac{x}{1 + t} & (0 \leq x < 1 + t) \\
1 & (x \geq 1 + t)
\end{cases} \quad (38a)
\]

*Case 2*: Likewise,

\[
\rho = \begin{cases} 
0 & (x < 0) \\
\frac{x}{1 - t} & (0 \leq x < 1 - t) \\
-1 & (x \geq 1 - t)
\end{cases} \quad (38b)
\]

The solution Eq. (38b) breaks down at \( t = 1 \); as the sketch on the hand-out shows, the characteristics intersect at \( t = 1 \) in Case 2 and the profile of \( \rho \) against \( x \) becomes triple-valued (but this cannot occur in reality).
### 3 Model of traffic flow

We assume:

1. One lane of traffic in direction of $Ox$ with no overtaking.

2. We can define a local car density $\rho = \rho(x, t)$ as the number of cars per unit length of road.

3. The local car velocity $v(x, t)$ is a function of $\rho$ alone, i.e.

\[
v = v(\rho)
\]  

(39)

The meaning of Eq. (39) is that each driver adjusts his, her or its speed to local conditions exclusively, whereas most drivers look ahead and adjust speed where appropriate. These assumptions give a car flowrate $q(\rho)$ with

\[
q(\rho) = \rho v(\rho)
\]  

(40)
Consider two values of $x$, viz. $x_1, x_2$ with $x_1 \leq x \leq x_2$.

At time $t$, the number of cars in this interval is

$$
\int_{x_1}^{x_2} \rho(x, t) \, dx.
$$

The rate of change of this must be the net flowrate, viz.

$$
\frac{\partial}{\partial t} \left\{ \int_{x_1}^{x_2} \rho(x, t) \, dx \right\} = [q(x, t)]_{x_1}^{x_2}
$$

(41)
III Traffic Flow Problem

If $x_1 = x$, $x_2 = x + \delta x$, Eq. (41) becomes

$$\frac{\partial}{\partial t} \rho \delta x = -\frac{\partial q}{\partial x} \delta x$$

⇒

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0.$$  (42)

• We need to model $v(\rho)$.

We assume there is a maximum possible density $P$ with essentially “bumper-to-bumper” traffic. When $\rho = P$, we assume $v(\rho)$ in Eq. (39) is zero.

We also assume $v(\rho)$ decreases as $\rho$ increases, with a maximum of $V$ when $\rho = 0$.

These assumptions are shown schematically...
III Traffic Flow Problem

With Eq. (40), Eq. (42) becomes

\[ \frac{\partial \rho}{\partial t} + \frac{d}{d \rho}(\rho v(\rho)) \frac{\partial \rho}{\partial x} = 0 \]

or

\[ \frac{\partial \rho}{\partial t} + c(\rho) \frac{\partial \rho}{\partial x} = 0, \]

(43)

\[ c(\rho) = \frac{d}{d \rho}(\rho v(\rho)) = v(\rho) + \rho v'(\rho) \]
The assumptions made about $v(\rho)$ give a $c(\rho)$ which is monotonic decreasing and negative for $\rho/P > \alpha$, where $\alpha$ is the value of $\rho/P$ for which $q(\rho)$ is a maximum.
3.1 Small amplitude disturbances from a uniform state

- Before studying the full non-linear problem, it is instructive to consider a simpler one. Suppose that there is almost a uniform state with $\rho = \rho_0$ and

$$\rho = \rho_0 + \rho' \text{ with } |\rho'| \ll \rho_0. \quad (44)$$

Linearise Eq. (43) - as with sound waves earlier - to get

$$\frac{\partial \rho'}{\partial t} + c(\rho_0) \frac{\partial \rho'}{\partial x} = 0. \quad (45)$$

Either

$$\frac{dt}{1} = \frac{dx}{c(\rho_0)} = \frac{d\rho'}{0} \Rightarrow$$

$$\rho' = \text{const. on } x - c(\rho_0)t = \text{const.}$$

Or put $\xi = x - c(\rho_0)t \Rightarrow$

$$\frac{\partial \rho'}{\partial x} = \frac{\partial \rho'}{\partial \xi}, \quad \left(\frac{\partial \rho'}{\partial t}\right)_x = \left(\frac{\partial \rho'}{\partial t}\right)_\xi - c(\rho_0) \left(\frac{\partial \rho'}{\partial \xi}\right) \Rightarrow$$

$$\left(\frac{\partial \rho'}{\partial t}\right)_\xi = 0.$$ 

Thus the GS of Eq. (45) is

$$\rho' = f \{x - c(\rho_0)t\}. \quad (46)$$
The characteristics of Eq. (45) are the straight lines

\[ x = \xi + c(\rho_0) t. \]  

(47)

Eq. (46) shows that \( \rho' \) is constant on each characteristic.

Eq. (46) represents a wave travelling to the right with speed \( c(\rho_0) \). If \( \rho_0/P > \alpha \Rightarrow c(\rho_0) < 0. \)

This is a kinematic wave; \( c(\rho_0) \) is the speed of the disturbance, not of the cars.

This explains a common phenomenon on a busy road when a sudden increase in density reaches you from ahead with no apparent reason.
3.2 The initial value problem for Eq. (43)

We wish to solve Eq. (43) subject to the initial condition

\[ \rho(x, 0) = f(x) \quad (48) \]

By the earlier methods - see especially the Example in § (5.2) - \( \rho \) is constant on the characteristics

\[ \frac{dt}{1} = \frac{dx}{\rho} \Rightarrow \frac{dx}{dt} = \rho. \]

Since \( \rho \) is constant on a characteristic, the characteristics are straight.

Thus, if \( c \{ f(\xi) \} = F(\xi) \), the solution can be written for \( t \geq 0 \)

\[ \rho = f(\xi) \quad \text{on the straight line} \quad x = \xi + F(\xi)t. \quad (49) \]
III Traffic Flow Problem

Example

Suppose
\[ v(\rho) = \frac{V}{P}(P - \rho) \quad (50) \]
and that \( \rho(x, 0) = f(x) \) satisfies
\[ \rho = \frac{1}{2}(\rho_L + \rho_R) - \frac{1}{2}(\rho_L - \rho_R) \tanh \frac{x}{L} \quad (51) \]
where \( \rho_L, \rho_R \) and \( L \) are constants. Discuss the solution given by Eq. (49) when

(i) \( \rho_L > \rho_R \)
and
(ii) \( \rho_L < \rho_R \).

Solution

From flow model Eq. (50) it follows

![Graph](image-url)

Figure 14: (a) Car flow \( v(\rho) = V(P - \rho)/P \)
\[ q(\rho) = \frac{V}{P} \left( P\rho - \rho^2 \right), \quad c(\rho) = \frac{V}{P}(P - 2\rho). \quad (52) \]

Note also that

\[ 0 < \frac{\rho}{c} < \frac{1}{4} \]

Figure 12: (b) Car flow flux, and (c) speed of disturbance

\[ \rho \rightarrow \rho_L \quad \text{as} \quad \frac{x}{L} \rightarrow -\infty \]

and

\[ \rho \rightarrow \rho_R \quad \text{as} \quad \frac{x}{L} \rightarrow +\infty. \]

Also

\[ F(\xi) = c \{ f(\xi) \} \]

\[ = \frac{V}{P} \left[ P - \rho_L - \rho_R + (\rho_L - \rho_R) \tanh \left( \frac{\xi}{L} \right) \right] \quad (53) \]
III Traffic Flow Problem

Case (i): $\rho_L > \rho_R \Rightarrow F'(\xi) > 0$ for $\forall \xi$

Figure 13: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (i). Note, that $\rho$ is constant on each characteristic.
Case (ii): $\rho_L < \rho_R \Rightarrow F'(\xi) < 0$ for $\forall \xi$

Figure 14: (a) Car flow, (b) profile of speed of disturbance, and (c) characteristics for Case (ii). Note, from (c) that characteristics eventually intersect

Characteristics eventually intersect $\Rightarrow$ problem becomes ill-posed.
III Traffic Flow Problem

If two characteristics intersect, any enclosed characteristic must meet one of them at an earlier time ⇒ earliest intersection must be between neighbouring characteristics.

Figure 15: Intersecting characteristics

Suppose these are

\[
x = \{\xi + F(\xi)t\}
\]

\[
x = \{(\xi + \partial\xi) + F(\xi + \partial\xi)t\}
\]

\[
= \{\xi + F(\xi)t\} + \{1 + F'(\xi)t\} \partial\xi
\]

\[\therefore 1 + F'(\xi)t = 0. \quad (54)\]

We get solutions of Eq. (54) with \( t > 0 \) only if \( \exists \xi \) with \( F'(\xi) < 0 \). [Thus for \( \rho_L > \rho_R \) there are no intersections and the solution given by Eq. (49) applies for \( \forall t \geq 0 \).]
The first positive $t$ satisfying Eq. (54) occurs when

$$t = T_{\text{min}} = \frac{1}{\text{Max}_{-\infty < \xi < \infty} \{-F''(\xi)\}}.$$  \hspace{1cm} (55)

While Eqs. (54) and (55) are general, we can calculate $T_{\text{min}}$ in our particular case when Eq. (53) holds.

We find

$$-F''(\xi) = \frac{V}{P}(\rho_R - \rho_L)\frac{1}{L}\text{sech}^2\left(\frac{\xi}{L}\right)$$

$$\Rightarrow$$

$$\text{max} \{-F''(\xi)\} = \frac{V(\rho_R - \rho_L)}{PL}$$

when $\xi = 0$. Then Eq. (55) gives

$$T_{\text{min}} = \frac{PL}{V(\rho_R - \rho_L)}. \hspace{1cm} (56)$$
3.3 Shocks

- We can understand in another way why there is trouble when $\rho_L < \rho_R$. With $c'(\rho) < 0$, low densities propagate forward relative to high densities. The profile of $\rho$ against $x$ inevitably steepens as $t$ increases and has a vertical section at $t = T_{\text{min}}$. Were we to continue, the profile would develop the triple-valued shape - clearly unacceptable since $\rho$ must be a single valued quantity.

- Instead the wave breaks, and the model must be extended. A consistent extension conserves cars but allows discontinuities in $\rho$ to occur across a shock.

Figure 16: Development of shock
III Traffic Flow Problem

We cannot have characteristics crossing one another. Instead the picture is as shown schematically.

Figure 17: Shock front and characteristics

Figure 18: Quantities at a shock front
III Traffic Flow Problem

From Eq. (41) ⇒

\[
\frac{\partial}{\partial t} \int_{x_1}^{s(t)} \rho \, dx + \frac{\partial}{\partial t} \int_{s(t)}^{x_2} \rho \, dx = q_1 - q_2
\]

\[
\text{LHS} = \int_{x_1}^{s(t)} \frac{\partial \rho}{\partial t} \, dx + \frac{1}{\partial t} \left\{ \int_{x_1}^{s(t)} - \int_{x_1}^{s(t)} \right\} \rho \, dx
\]

→ 0 as \( x_1 \to s_- \)

\[
+ \int_{s(t)}^{x_2} \frac{\partial \rho}{\partial t} \, dx + \frac{1}{\partial t} \left\{ \int_{s(t)}^{x_2} - \int_{s(t)}^{x_2} \right\} \rho \, dx
\]

→ 0 as \( x_2 \to s_+ \)

\[
= \frac{1}{\partial t} \left\{ \int_{s(t)}^{s(t) + \delta t} \rho_1 \, dx - \int_{s(t)}^{s(t) + \delta t} \rho_2 \, dx \right\}
\]

\[
= \dot{s} (\rho_1 - \rho_2)
\]

by the mean value theorem.

⇒

\[
\dot{s} (\rho_1 - \rho_2) = (q_1 - q_2) \quad (57a)
\]

\[
\dot{s} = \frac{q_1 - q_2}{\rho_1 - \rho_2}. \quad (57b)
\]
III Traffic Flow Problem

• The position of the shock is fixed by the need to conserve cars. Without a shock the curve of $\rho$ against $x$ would become (unacceptably) triple-valued. We insert the shock so that the shaded areas are equal, thus ensuring that $\int \rho \, dx = \text{number of cars is unchanged}. This is known as (Whitham’s) equal area rule.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure19}
\caption{Shock fitting: Whitham’s equal area rule}
\end{figure}

• As an application consider what happens as cars approach a stationary queue behind a red traffic light so that $\rho_R = P$, $\rho_L < P$. On meeting the queue cars stop and the lengthening of the queue is achieved by a shock wave propagating backwards.
3.4 The Riemann problem

- We wish to consider the case when we solve Eqs. (43) and (48) where there is a discontinuity in $f(x)$. It will be sufficient to consider the simplest possible case, viz.

$$\rho(x, 0) = f(x) = \begin{cases} 
\rho_L & (x < 0) \\
\rho_R & (x > 0) 
\end{cases} \quad (58)$$

- Then Eq. (49) gives the solution as

$$\rho = \rho_L \quad \text{on} \quad x = \xi + c(\rho_L) t \quad (\xi < 0)$$

$$\rho = \rho_R \quad \text{on} \quad x = \xi + c(\rho_R) t \quad (\xi > 0) \quad (59)$$

Consider first the case $\rho_L > \rho_R$. The characteristic diagram is easy to draw...
They are either parallel to $OA$ with slope $c(\rho_L)$; to the left of $OA$, $\rho = \rho_L$. Or they are parallel to $OB$ with slope $c(\rho_R)$; to the right of $OB$, $\rho = \rho_L$. But what happens in $OAB$?

• The problem arises because of the discontinuity and can be solved by considering a limit process in which $\rho$ takes all the values from $\rho_R$ to $\rho_L$, and all the characteristics go through the origin. Thus

$$\rho = k \quad (\rho_R < \rho < \rho_L)$$

on $x = c(k)t$.

The solution is therefore:
III Traffic Flow Problem

\[
\rho = \begin{cases} 
\rho_L & : \quad x < c(\rho_L) t \\
\text{k on } x = c(k)t : \quad c(\rho_L) < c(k) < c(\rho_R) & (60) \\
\rho_R & : \quad x > c(\rho_R) t
\end{cases}
\]

Figure 21: Centered fan or expansion fan corresponding or rarefication wave

The characteristic diagram is augmented by a centered fan or an expansion fan or expansion wave or rarefication wave.
Conversely, when $\rho_L < \rho_R$ the characteristic diagram shows immediately trouble whose only resolution is a shock starting from $t = 0$ with speed, given by Eq. (57b) as $U$, where

$$U = \frac{q(\rho_L) - q(\rho_R)}{\rho_L - \rho_R}$$

(61)

Figure 22: Schematic shock with speed $U$
3.5 Additional refinements

• The model assumptions leading to Eq. (43) are too simple. One extension is to suppose that \( q \) is a function of the density gradient \( \partial \rho / \partial x \) as well as \( \rho \), thus allowing drivers to reduce their speed to account for an increasing density ahead. A simple assumption is to take

\[
q = Q(\rho) - \nu \rho_x
\]

where \( \nu \) is a positive constant. Thus \( q \) decreases if \( \rho_x \) is positive, i.e. it there is an increasing density ahead. Use of Eq. (62) in Eq. (42) gives

\[
\rho_t + c(\rho)\rho_x = \nu \rho_{xx}, \quad c(\rho) = q'(\rho)
\]  

(63)

• Seek solutions of Eq. (63) of the form

\[
\rho = \rho(X) \quad X = x - Ut
\]

(64)

where \( U \) is a constant still to be determined. Substitution in Eq. (63) \( \Rightarrow \)

\[
-U \rho'(X) + c(\rho)\rho'(X) = \nu \rho''(X).
\]

Since \( c(\rho) = Q'(\rho) \) we have
\[ Q(\rho) - U\rho + C = \nu\rho'(X) \] (65)

where \( C \) is a constant. Suppose \( \rho \to \rho_L \) as \( X \to -\infty \) and \( \rho \to \rho_R \) as \( X \to +\infty \) ⇒

\[ Q(\rho_L) - U\rho_L + C = Q(\rho_R) - U\rho_R + C = 0, \]

⇒

\[ U = \frac{Q(\rho_R) - Q(\rho_L)}{\rho_R - \rho_L} \] (66)

This is exactly Eq. (57b) but (for the moment) in a different context.

• Since \( \rho \to \rho_L \) as \( X \to -\infty \) and \( \rho \to \rho_R \) as \( X \to +\infty \), \( \rho'(x) = 0 \) at \( \rho = \rho_L \) and \( \rho = \rho_R \). We suppose \( \rho_L \) and \( \rho_R \) are simple zero’s of

\[ Q(\rho) - U\rho + C, \]

and more precisely we shall suppose \( \rho_L < \rho_R \) and

\[ Q(\rho) - U\rho + C = \alpha (\rho - \rho_L)(\rho_R - \rho) \quad (\alpha > 0) \] (67)
With $\alpha > 0$,

$$c(\rho) = Q'(\rho) = \alpha (\rho_R - \rho) - \alpha (\rho - \rho_L)$$

and

$$c'(\rho) = \alpha (\rho_L - \rho_R) < 0.$$

We can always approximate $Q(\rho)$ by a quadratic. Then Eq. (65) becomes

$$\nu \frac{d\rho}{dX} = \alpha (\rho - \rho_L) (\rho_R - \rho)$$

with solution

$$\left(\frac{\rho_R - \rho}{\rho - \rho_L}\right) = \left(\frac{\rho_R - \rho_0}{\rho_0 - \rho_L}\right) e^{-\frac{X}{L}}, \quad L = \frac{\nu}{\alpha (\rho_R - \rho_L)}$$

(68)

where $\rho = \rho_0$ at $X = 0$. We note that $\rho \to \rho_L$ as $X \to -\infty$ and that $\rho \to \rho_R$ as $X \to +\infty$ as required.
The transition between $\rho \sim \rho_L$ and $\rho \sim \rho_R$ occupies a thickness of order $L$.

As $L$ diminishes, i.e. as $\nu$ diminishes for fixed $\alpha$ and $(\rho_R - \rho_L)$ the transition takes place more sharply $\Rightarrow$ a shock is approached.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{shock_front.png}
\caption{Development of shock front}
\end{figure}

- The model in this section can be taken further in the case when Eq. (67) holds.
Multiply Eq. (63) by \( c'(\rho) \Rightarrow \)

\[
c'(\rho) \rho_t + c(\rho)c'(\rho) \rho_x - \nu c'(\rho) \rho_{xx}
\]

because

\[
\frac{\partial c}{\partial x} = c'(\rho) \frac{\partial \rho}{\partial x}
\]

and therefore

\[
\frac{\partial^2 c}{\partial x^2} = c''(\rho) \left( \frac{\partial \rho}{\partial x} \right)^2 + c'(\rho) \frac{\partial^2 \rho}{\partial x^2}.
\]

In the case when \( Q(\rho) \) is quadratic, i.e. Eq. (67) holds, \( c''(\rho) = 0 \) since \( c(\rho) = Q'(\rho) \). Thus

\[
c_t + cc_x = \nu c_{xx}.
\]  

(69)

This is known as Burger’s equation and, remarkably, it can be solved explicitly by means of the transformation

\[
c = -2\nu \frac{\phi_x}{\phi}
\]  

(70)

discovered independently by E. Hopf (1950) and J.D.Cole (1951). Use of Eq. (70) transforms Eq. (69) into the standard linear equation (after one integration w.r.t. x):

\[
\phi_t = \nu \phi_{xx}.
\]  

(71)

It can be shown that this is also consistent with the shock structure.
A second refinement is that there is a time lag in driver response. One way of handling this is to take Eq. (62) and deduce from it that \( v = q/\rho \) satisfies

\[
v = V(\rho) - \frac{\nu}{\rho} \rho_x, \quad V(\rho) = \frac{Q(\rho)}{\rho}.
\]  

(72)

Then regard this as a velocity which the driver tries to achieve. The acceleration of the car is

\[
\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} 
\]

[see Notes after § (4.3)] and the model is

\[
v_t + vv_x = -\frac{1}{\tau} \left\{ v - V(\rho) + \frac{\nu}{\rho} \rho_x \right\},
\]

(73)

where \( \tau \) is a measure of the response time. Eq. (73) is to be solved together with Eq. (42), i.e.

\[
\rho_t + (\rho v)_x = 0.
\]

(74)
4 1-D Linear convection-dominated problems

4.1 1-D linear convection equation

\[ \frac{\partial \bar{T}}{\partial s} + u \frac{\partial \bar{T}}{\partial x} = 0 \] (75)

- Schemes: FTCD, Upwind Differencing, Leapfrog, lax-Wendroff, Crank-Nicolson ⇒ Handout pp.278-279

Observe: stencil of schemes, algebraic form, leading term of truncation error, and stability.
III Traffic Flow Problem

- Example: linear convection of a truncated sine wave

Solution of Eq. (75) subject to IC/BC:
\[
\begin{align*}
\bar{T}(x, 0) &= \sin(10\pi x) \quad \text{for } 0 \leq x \leq 0.1, \\
&= 0 \quad \text{for } 0.1 < x \leq 1.0, \\
\bar{T}(0, t) &= 0 \quad \text{and } \bar{T}(1, t) = 1.
\end{align*}
\]

Note: Observe dissipation (Figs. 9.2-9.4), dispersion (slow-down) and oscillatory wake (Figs. 9.3-9.4).
Fig. 9.4. Crank-Nicolson finite difference solution for the convection equation with $C=0.8$
4.2 Numerical dissipation and dispersion

-Most (M)HD phenomena are governed by hPDEs, they contain no or little dissipation ⇒

-Solutions are characterised by wave-trains that propagate with little or no loss of amplitude ⇒

-Numerical schemes ‘should not’ introduce non-physical dissipation, and non-physical dispersion (see Figs. 9.2-4).

Solution for propagating plane wave subject to dissipation & dispersion

\[
\bar{T} = R T_{amp} e^{-p(m)t} e^{im[x-q(m)t]} \quad (76)
\]

Here \( p(m) \): amplitude attenuation; \( q(m) \): propagation speed. It’s instructive to consider two related eqs:

\[
\frac{\partial \bar{T}}{\partial t} + u \frac{\partial \bar{T}}{\partial x} - \alpha \frac{\partial^2 \bar{T}}{\partial x^2} = 0 \quad (77)
\]

\[
\frac{\partial \bar{T}}{\partial t} + u \frac{\partial \bar{T}}{\partial x} + \beta \frac{\partial^3 \bar{T}}{x^3} = 0 \quad (78)
\]
III Traffic Flow Problem

Eq. (77) is the transport equation, for plane waves:

\[ p(m) = \alpha m^2, \quad q(m) = 0 \]  
(79)

amplitude attenuated by diff term but propagation speed unaffected.

Eq. (78) is the linear Korteweg-de Vries eq. For plane waves:

\[ p(m) = 0, \quad q(m) = u - \beta m^2 \]  
(80)
amplitude unaltered but propagation speed depends on wavelength.

⇒:

- dissipation → attenuation ↔ even-ordered spatial derivatives;

- dispersion → propagation of waves with different wave number \( m \) at different speeds \( q(m) \) ↔ odd-ordered spatial derivatives.

⇒ Through discretisation the truncation error consist, typically, of higher even- and odd-ordered derivatives!
III Traffic Flow Problem

5 1-D Burger’s Equation

5.1 Useful properties

Burger’s (1948)

\[
\frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} - \nu \frac{\partial^2 \bar{u}}{\partial x^2} = 0 \tag{81}
\]

- Similar to transport equation, except the convective term is nonlinear
- If \( \nu = 0 \): inviscid Burger’s equation
- Effect of viscosity: (a) reduces amplitude; (b) prevents multi-valued solutions ⇒ Burger’s eq is very suitable for testing comp algorithms
- Cole-Hopf transformation allows exact solution for a wide range of IC/BCs
- Alternative way to handling nonlinear convective form by using conservation form:

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{F}}{\partial x} - \nu \frac{\partial^2 \bar{u}}{\partial x^2} = 0, \quad \bar{F} = \frac{1}{2} \bar{u}^2 \tag{82}
\]

Note: Aliasing can be a problem in astro/geophysical problems where dissipation is generally small and cannot reduce errors caused by energy reappearance from scales \( \leq 2\Delta x \) to longer wavelength.
5.2 Explicit schemes

- FTCS finite difference rep

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} - \frac{\nu(u_{j-1}^n - 2u_j^n + u_{j+1}^n)}{\Delta x^2} = 0
\]

The contribution \( u_j^n \) to convective term is the local solution at node \( j, n \). Since this is available directly the various schemes mentioned before can be applied to BE.

- FTCS of the conservative form (Eq. 82)

\[
\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F_{j+1}^n - F_{j-1}^n}{2\Delta x} - \frac{\nu(u_{j-1}^n - 2u_j^n + u_{j+1}^n)}{\Delta x^2} = 0
\]

A potentially more accurate treatment of the convective term is provided by a four-point upwind discretisation, e.g. the truncation error is only \( O(\Delta x^2) \).

- Lax-Wendroff scheme

Less economic because evaluation at half-steps \( (j - \frac{1}{2}, j + \frac{1}{2}) \) is involved.
5.3 Implicit schemes

Note: It is *not* so straightforward as for linear equations.

- Crank-Nicolson formulation

\[
\frac{\Delta u_{j}^{n+1}}{\Delta t} = -\frac{1}{2}L_{x}(F_{j}^{n} + F_{j}^{n+1}) + \frac{1}{2}\nu L_{xx}(u_{j}^{n} + u_{j}^{n+1})
\]

where

\[
\Delta u_{j}^{n+1} = u_{j}^{n+1} - u_{j}^{n}; L_{x} = \frac{(-1, 0, 1)}{2\Delta x}; L_{xx} = \frac{(1, -2, 1)}{\Delta x^{2}}.
\]

Problem: Difficult to reduce to a sys of lin tridiagonal eqs for the solution \(u_{j}^{n+1}\) and to use the very efficient Thomas algorithm because of the nonlin implicit term \(F_{j}^{n+1}\).

Solution: introduce a correction term with split scheme! (I.e., Taylor series expansion of \(F_{j}^{n+1}\) is made about the \(n\)th time level resulting in a tridiagonal logarithm)

\[
\frac{\Delta u_{j}^{n+1}}{\Delta t} = -\frac{1}{2}L_{x}(2F_{j}^{n} + u_{j}^{n} \Delta u_{j}^{n+1}) + \frac{1}{2}\nu L_{xx}(u_{j}^{n} + u_{j}^{n+1})
\]

Here the truncation error: \(O(\Delta t^{2}, \Delta x^{2})\) and unconditionally stable in the von Neumann sense.